ESTIMATING BED SHEAR FROM VELOCITY PROFILE

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A graphical method of estimating bed shear from measured velocity profiles is presented as an alternative to logarithmic law approach. In the present approach the entire velocity profile is considered as per binary law of velocity distribution i.e., logarithmic law in the wall region and parabolic law in the outer region. The validity of this method has been demonstrated for a typical velocity profile. An analysis has been also made in case of an erroneous measurement of bed level.

KEY WORDS: Bed Shear, Boundary Effects, Logarithmic Profile, Parabolic Laws, Velocity Profile.

Introduction

The usual method of estimating the bed shear is based on the measurements of velocity profile or relevant turbulent intensities of flow. For two-dimensional, steady, and quasi-steady uniform turbulent flow, it is well recognized that the mean velocity follows a logarithmic profile in the region above the buffer region (Nikora et al., 2001; Song et al., 1994; Ferro and Baiamonte, 1994; Cardoso et al., 1989; Einstein and El-Samni, 1949). Consequently, the bed shear is often estimated from measurements of the velocity profile (Balachandar et al., 2002; Bergeron and Abrahams, 1992; Whiting and Dietrich, 1990; Wang and Qian, 1989).

Local bed shear can be computed from the logarithmic relation between the shear velocity and the variation of velocity with height (Schlichting, 1987):

\[
\frac{u}{u_*} = \frac{1}{\kappa} \ln \left( \frac{Y}{Y_0} \right),
\]

where \( u \) is velocity, \( u_* \) – shear velocity \( = \left( \frac{\tau_0}{\rho} \right)^{0.5} \), \( \kappa \) – the von-Karman’s universal constant, \( Y \) – height above the bed and \( Y_0 \) – the characteristic roughness length. Although this method is widely used (Wilcock, 1996; Robert, 1997; Biron et al., 1998; Lawless and Robert, 2001; Babaeyan-Koopaei et al., 2002), there are many uncertainties in fitting a logarithmic profile to velocity data (Wilkinson, 1984; Dietrich and Whiting, 1989; Williams, 1995). A single law of velocity distribution may not describe satisfactorily the entire velocity profile of varying effects of boundary on the velocities at different distances from the velocity (Rao and Kumar, 2006). It is generally understood (Hinze, 1975) that velocity profile in the wall region (excluding a thin viscous layer closest to the boundary wall) is logarithmic in nature whereas it is parabolic in the outer region (Vedula and Rao, 1985). It is a general practice to use logarithmic law for the estimation of bed shear, which can be obtained from the slope of a straight line plot of velocities versus logarithmic of the corresponding distances measured from the boundary. However, apportionment of flow region
may be not possible before hand. So, it is likely that such straight lines fits are made to the point extending beyond the wall region whereas logarithmic law is no more valid thus leading to erroneous estimates of bed shear. Another local estimate of bed shear uses the quadratic stress law which relates the average shear at the bed to the square of the average fluid velocity \( (U_0) \) (Schlichting, 1987):

\[
\tau_0 = \rho C_d U_m^2,
\]

where \( C_d \) is the drag coefficient (Williams, 1995). The main difficulty with this technique is that the drag coefficient is not a constant and it is therefore difficult to estimate it accurately (Dietrich and Whiting, 1989).

This paper presents a new method to determine bed shear with reasonable accuracy. In this method, the entire velocity is taken into consideration by assuming logarithmic law in the wall region and parabolic law in the outer region and concepts of binary law (Vedula and Rao, 1985) connecting those two laws are used.

**Theoretical considerations**

The velocity distribution in the inner zone of a boundary layer can be described by the log law, and that of the outer zone can be described by the parabolic law. At the common boundary of these two zones, both laws are valid, and the velocity is single valued. This condition is satisfied only if the constants in the two laws are related in a particular way. If a further condition is specified that at the junction, the slopes of the two curves must be the same – i.e., the velocity distribution curves should meet tangentially – one more relationship between the constants is obtained. The binary law of velocity distribution (Vedula and Rao, 1985) can be expressed as:

\[
\frac{U-u}{u_*} = -\frac{1}{\kappa} \ln \left( \frac{y}{D} \right) + B_* \quad \text{for } 0 < \frac{y}{D} < x \quad (3a)
\]

\[
\frac{U-u}{u_*} = C \left( 1 - \frac{y}{D} \right)^2 \quad \text{for } x < \frac{y}{D} < 1 \quad (3b)
\]

where \( B_* \) is a correction term for velocity defect log-law, \( C \) – a coefficient for parabolic law, \( u \) – the velocity at distance \( y \) from the boundary, \( U \) – the maximum velocity at \( y = D \) (the flow depth), \( u_* \) – bed shear velocity, \( \kappa \) – the Von-Karman’s universal constant, \( x \) is a value of \( y/D \) up to which logarithmic law is valid and beyond \( x \), the parabolic law is valid. It may be called as ‘depth of deviation point’. As the velocity profile is continuous, at depth of deviation point, the velocities obtained from both law should be equal; so also their slopes. Thus expressions of \( B_* \) and \( C \) can be obtained as:

\[
B_* = \frac{1}{\kappa} \left( \frac{1}{2x} \right) + \frac{1}{\kappa} \ln (x)
\]

\[
& C = \frac{1}{\kappa} \left( \frac{1}{2x(1-x)} \right)
\]

It may be mentioned here that the binary law of velocity distribution as expressed above, is found to be valid (Vedula and Rao, 1985) for plat plates, pipes and for wide open channels. Interestingly it is found that the value of ‘\( x \)’ varies with the type of conveyance systems, flow conditions and roughness of the boundary etc. More detailed discussion on estimation of ‘\( C \)’ and ‘\( x \)’ can be found in Vedula and Rao (1985).

**Estimation of bed shear velocity**

A theoretical plot of binary law of velocity distribution is shown in Fig. 1. The straight line AD represents parabolic law in the outer region (Eq. 3b)) and curve DQ represents logarithmic law in the inner region as per Eq. (3a)).

The point D corresponds to the depth of deviation point separating wall and outer regions. The straight line AD is extended to cut the velocity axis at \( P \), as shown in Fig. 1. A vertical line is drawn from \( P \) cutting the logarithmic axis at \( Q \). ‘QRS’ is a horizontal line cutting the extended straight line at \( R \) and the ordinate at \( S \).

Let \( AS = Z \). Then as per Eq. (3a))

\[
QS = -\frac{1}{\kappa} u_* \ln \left( 1 - \sqrt{Z} \right) + B_{site}
\]

and as per Eq. (3b)

\[
OP = C_{site} \quad \text{and } RS = C u_* Z.
\]

As \( OP = QS \), \( Z \) can be expressed as:

\[
Z = \left[ 1 - \exp \left( (B_* - C) \kappa \right) \right]^2.
\]

The intercept, \( QR = QS - RS = C u_* (1 - Z) \).

Or, say \( QR = F u_* \) [where, \( F = C (1 - Z) \)].

It may be noted that \( Z \) is purely a function of \( x \), whereas \( F \) is also a function of \( x \) considering \( \kappa \) as
Fig. 1. Binary law of velocity distribution.
Obr. 1. Binárný zákon rozdelenia rýchlostí.

universal constant. The values of $Z$, $F$ and $F + \sqrt{Z}$ with $\kappa = 0.408$ for different values of $x$ ranging from 0.15 to 0.30 are shown in Fig. 2. This range is considered to be a practical range for most conveyance systems.

The following useful deductions for the bed shear estimation can be based on the above mentioned theoretical and graphical analyses binary law of velocity distribution:

A. It can be seen from Fig. 2 that for a wide variation of $x$ from 0.15 to 0.3, the function $F$ varies only from 0.946 to 0.993, that is variation in $F$ is only up to ± 2.5% about a mean value of 0.97. Thus form the intercept $QR$ (Fig. 1) may be taken as equal to 0.97 $u^*$, thus $u^*$ can be determined.

B. If the precise value of $Z$ is known, $u^*$ can be computed accurately. By knowing $Z$, $x$ can be determined by trial and error solution of Eq. (7) in which $B_*$ and $C$ are known in terms of $x$. From the binary law plot $Cu^*$ is known and hence $u^*$ can be computed.

C. There is yet another useful deduction that can be made use for shear estimation. In the Fig. 2, one can notice that the value of $F + \sqrt{Z}$ is fairly constant and it may be taken as 1.9. By knowing $Z$ from binary law plot, $F$ and $C$ can be computed and hence $u^*$ can be computed.

The above analysis of binary law is valid only when the so called ‘theoretical bed level’ is known. Like logarithmic approach, the present method also
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does not yield satisfactory estimates of \( u_* \), if there are errors in bed level estimation.

Graphical procedure as described above can be presented in an analytical approach as follows:
- Extrapolate the parabolic law to \( y = 0 \) where \( U - u = Cu_* \).
- Estimate height \( y_i \) where \( U - u = Cu_* \) using Eq. (3a), resulting in \( y_i = De^{(B - C)} \) or
  \[
  Z = \left( 1 - e^{B - C} \right)^2.
  \]
- Estimate difference of velocity estimates from Eqs. (3a) and (3b) at height \( y_i \). Thus it follows from \( (U - u_{log} - U + u_{par}) = u_{par} - u_{log} = Cu_*(1 - Z) \) and \( (u_{par} - u_{log})/u_* = C(1 - Z) = F \)
- \( u_{par} \) = velocity estimate from parabolic law and \( u_{log} \) = velocity estimate from log-law.

Verification of the model

A typical velocity profile measured by Nikuradse (1950) in sand roughened pipes is used for the purpose of illustration of present method. Two set of measurements have been taken to analyze the present method. In the first case, pipe radius is 2.462 cm, \( r/k_s \), (the sand roughness height) is 252 and Reynolds number of the flow is 21400. Second set comprises of respective values 4.97 cm, 504 and 22700. Besides that, measurements taken in a sand bed channel by Mahmood et al. (1980) and measurement in gravel bed by Ferro (2003) are also verified through the present model. It may be noted that Ferro (2003) has used Acoustic Doppler Velocimeter for the measurement of velocity profile. They are plotted in the Figs 3a) to 3d).

Fig. 3a) Velocity distribution in rough pipes \((r/k_s = 252)\).
Obr. 3a) Profíly rýchlostí v drsnej rúre \((r/k_s = 252)\).

Fig. 3b) Velocity distribution in rough pipes \((r/k_s = 504)\).
Obr. 3b) Profíly rýchlostí v drsnej rúre \((r/k_s = 504)\).
Fig. 3c) Velocity distribution in sand bed channel.
Obr. 3c) Rozdelenie rýchlostí v koryte z piesku.

Fig. 3d) Velocity distribution in gravel bed flume.
Obr. 3d) Rozdelenie rýchlostí v koryte zo štrkopiesku.

From Figs 3a) to 3d), the following observations can be made. The intercept, $F_* u = 2.8 \text{ cm sec}^{-1}$, $C_* u = 17.1 \text{ cm sec}^{-1}$ and $Z = 0.83$ for case 1, for case 2 corresponding values are $1.44 \text{ cm sec}^{-1}$, $10.16 \text{ cm sec}^{-1}$ and $0.85$, for the third case, $F_* u = 6.3828 \text{ cm sec}^{-1}$, $C_* u = 41.3309 \text{ cm sec}^{-1}$ and $Z = 0.839$ and for the 4th case, $F_* u = 8.488 \text{ cm sec}^{-1}$, $C_* u = 51.1534 \text{ cm sec}^{-1}$ and $Z = 0.822$.

The values of $u_*$ can be computed based on three deductions given earlier and they are tabulated in the Tab. 1.

As shown in the Tab. 1, prediction error is well below of 10%. It ranges between 0.3 to 8%. Thus it can be said that the present method is giving a reasonable estimate considering the various factors involved during the measurements of velocity profile.

In the analysis of data, in the core region, as in Figs 3a) to 3d), a slight deviation was observed from the straight line at and in the immediate neighborhood of the axis of the pipe (in the region $0.98 u/U \leq 1$). This may be due to the reason that the concept of constant eddy viscosity, used in deriving the parabolic law, is no more valid in the neighborhood of the pipe axis. However, this small deviation does not warrant any correction as far as bed shear estimation is concerned since a straight line can be fitted easily in the outer region leaving out such deviation points.

**Error in estimating bed level**

Turbulent well developed flow conditions are governed not only by the geometric characteristic of the cross section of the flow, but, also, by the
shape, size and nature of the roughness elements present at the bottom of the river channel and by the mechanism of the sediment transport process. Thus the relative importance between the roughness length scale and the average water depth of the flow (Malaika et al., 1961; Perry et al., 1969; Becchi and Pedrizzetti, 1987; Pyle et al., 1981) seems well discriminate among the different flow regime. As said, the vertical variation of velocity with elevation above the bed has been used often to estimate bed shear stress (Bridge and Jarvis, 1976, 1977; Ashworth and Ferguson, 1986; Petit, 1990; Hassan and Reid, 1991; Robert et al., 1992). It may be difficult to obtain several velocity measurements in the portion of the flow where the law of the wall applies, i.e. the bottom 20 per cent of the flow depth (Bathurst, 1982; Nezu and Nakagawa, 1993). Application of the present model also needs the location of the bed surface datum for the profile must be specified.

Let us suppose that there are errors in measurement of \( D \) (–10 % to +10 %). This means that the instead of true bed level, we are taking measurement from erroneous bed level, in case of which the value of \( y/D \) becomes \( (y-\Delta D)/(D-\Delta D) \). This implies that there is going to a change in the deviation point and the new deviation point is \( x_{\text{new}} = x_{\text{old}} - E(1-E) \), where \( E = \Delta D/D \). This scenario is presented in the Fig. 4.

Present section analyses the effect of this change on the prediction of \( Z \) and \( F \), which in turn is going to be effect on the estimation of bed shear. Fig. 5a) and 5b) shows the prediction error of \( Z \) and \( F \) due to the erroneous measurement of bed level. As shown in Fig. 5a), maximum error on estimation of \( Z \) varies between –4.5 to 6.5%. Considering the difficulty lies in true bed level estimation, these ranges may be acceptable in estimating of \( Z \). And it can be assumed that the same range is going to affect the estimation of \( u_* \). The variation of error in estimation \( F \) is shown in the Fig. 5b).

<table>
<thead>
<tr>
<th>Measurement by</th>
<th>( u_* ) [cm sec(^{-1})]</th>
<th>( x )</th>
<th>( u_* ) calculations</th>
<th>Error variation [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nikuradse-1</td>
<td>2.87</td>
<td>0.3</td>
<td>2.89</td>
<td>2.93</td>
</tr>
<tr>
<td>Nikuradse -2</td>
<td>1.514</td>
<td>0.2995</td>
<td>1.392</td>
<td>1.466</td>
</tr>
<tr>
<td>Mahmood et al.</td>
<td>6.765</td>
<td>0.2433</td>
<td>6.58</td>
<td>6.213</td>
</tr>
<tr>
<td>Ferro</td>
<td>8.51</td>
<td>0.2996</td>
<td>8.75</td>
<td>8.76</td>
</tr>
</tbody>
</table>

Fig. 4. Error in the bed level measurement.
Obr. 4. Chyba v meraní dna.
As discussed earlier, the value of $F$ is having a direct effect on the measurement of $u_*$. And the error ranges in estimating the $F$ is in very narrow range, as shown in Fig. 5b), if the numerical value of the error introduced in the bed is taken into consideration (–10 to 10%). It varies from –2.5 to 3%. These values of errors have been calculated by assuming that if there were no error in the measurement of bed level. The value of $F$ and $Z$, when there was no error in the measurement of $D$, is shown in the Fig. 2.
Conclusion

Method for estimating bed shear from measured velocity profiles has been presented through the present work. This method assumes the binary law of velocity distribution. Present method fits very well with the literature data. And it can be also said that the present way of estimating bed shear through velocity profiles predicts the bed shear in a practical range, when there is an error in the measurement of true bed level.

List of symbols

- $B_0$ – correction factor,
- $C_d$ – drag coefficient,
- $C$ – constant,
- $D$ – depth of flow [m],
- $u$ – velocity [m s$^{-1}$],
- $E$ – error in the measurement of $D$,
- $U_m$ – average velocity [m s$^{-1}$],
- $u_c$ – shear velocity [m s$^{-1}$],
- $x = \frac{y}{D}$,
- $y$ – lateral distance from side wall [m],
- $Y_h$ – height above the bed [m],
- $y_0$ – the characteristic roughness length [m],
- $Z_0$ – value equal to $(1-y^2)^{-1/2}$,
- $\tau_0$ – shear Stress [N m$^{-2}$],
- $\rho$ – density [kg m$^{-3}$],
- $\kappa$ – von Karman constant.

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URČENIE TRENA NA STENÁCH KORYTA Z PROFILOV RÝCHLOSTI

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Grafická metóda určenia charakteristik trenia na dne koryta z meraných profilov rýchlostí je alternatívu k používanému prístupu vychádzajúcemu z logaritmic-kého zákona rozdelenia rýchlostí prúdeňa. Tento nový prístup predpokladá binárne rozdelenie rýchlostí prúdeňa. Sily trenia vypočítané navrhovanou metódou dávajú výsledky, ktoré veľmi dobre súhlasia s údajmi v literatúre.

Zoznam symbolov

$B_*$ – korekčný faktor,
$C_d$ – súčiniteľ odporu,
$D$ – hĺbka tekutiny [m],
$u$ – rýchlosť [m s$^{-1}$],
$E$ – chyba merania $D$,
$U$ – maximálna rýchlosť [m s$^{-1}$],
$U_m$ – priemerná rýchlosť [m s$^{-1}$],
$u_*$ – rýchlosť trenia (trecia rýchlosť) [m s$^{-1}$],
$x = y/D$,
$y$ – laterálna vzdušnosť od bočnej steny [m],
$Y$ – výška nad dnom [m],
$Y_0$ – charakteristická drsnosť [m],
$Z = (1-x)^2$,
$\tau_0$ – tangenciálne napätie [N m$^{-2}$],
$\rho$ – hustota [kg m$^{-3}$],
$\kappa$ – konštanta von Karmana.