Determination of hydraulic conductivity of a suspension

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Abstract: The process of sedimentation and subsequent gravity compression of kaolin and water suspensions was investigated experimentally. 45 batch tests were carried out and the time dependence of the height of the suspension column was measured. The one-dimensional equations of Darcian mechanics of two-phase porous media are applied to formulate the studied process mathematically. A very natural assumption makes it possible to find a solution of the forward problem for a starting period of the process. Analysis of the theoretical function and the experimental data gives hydraulic conductivity as a function of the suspension concentration. The obtained results are presented and discussed.

Keywords: Hydraulic conductivity; Kaolin; Suspension; Gravity thickening.

INTRODUCTION

Various industrial technologies require to study the process of suspension sedimentation and dewatering. According to its energetic efficiency, the gravitational thickening is one of the mostly utilized methods of lowering volume of suspensions and increasing the concentration of their solid phase. There are two kinds of standard laboratory tests, the batch tests and the continuous tests, which are utilized when examining the thickening process or when collecting data required for its mathematical modelling.

The presented research was aimed at carrying out a set of experiments with a well defined suspension of water and kaolin and to develop a convenient method of getting hydraulic conductivity of the suspension as a function of its concentration.

Numerous authors have presented contributions to experimental or theoretical study of suspension thickening processes. To cite the classical ones, we can start with (Coe and Cleveenger, 1916) and continue with well known paper by Kynch (1952) whose results were often adopted in later research. In our research, the degree of the suspension coagulation and gel point position were important and hence, we made use of the paper by Mls (1999). The presented research was aimed at carrying out a set of experiments with a well defined suspension of water and kaolin and to develop a convenient method of getting hydraulic conductivity of the suspension as a function of its concentration. The obtained results are presented and discussed.

Keywords: Hydraulic conductivity; Kaolin; Suspension; Gravity thickening.

THE APPLIED THEORY

According to Mls (1999), the general flow of both the solid phase and the liquid phase in one dimension is governed by following equations

\[
\frac{\partial n(x,t)}{\partial t}(x,t) + \frac{\partial w(x,t)}{\partial x} = 0, \quad \text{(1)}
\]

\[
\frac{\partial n(x,t)}{\partial t} - \frac{\partial v(x,t)}{\partial x} = 0, \quad \text{(2)}
\]

\[
\frac{\partial v(x,t)}{\partial t} + gn(x,t) + \frac{n(x,t) \frac{\partial p}{\partial x}}{\rho_w} + \frac{gn(x,t)}{K(x,t)}u(x,t) = 0, \quad \text{(3)}
\]

\[
\frac{\partial v(x,t)}{\partial t} + g(1 - n(x,t)) - \frac{1}{\rho_s} \frac{\partial \tau}{\partial x} + \frac{1 - n(x,t)}{\rho_s} \frac{\partial p}{\partial x} - \frac{gn(x,t)}{\rho_s K(x,t)}u(x,t) = 0, \quad \text{(4)}
\]

where \( n \) is porosity of the suspension, \( x \) is the space coordinate oriented vertically upwards, \( t \) is time, \( w \) and \( v \) are the liquid-phase and the solid-phase volumetric flux densities, respectively, \( g \) is gravity acceleration, \( \rho_w \) and \( \rho_s \) are the liquid-phase and the solid phase densities, respectively, \( \rho \) is the liquid-phase pressure, \( K \) is hydraulic conductivity of the suspension, \( \tau \) is the solid-phase stress and \( u \) is the relative liquid-phase volumetric flux density satisfying relation

\[
u = w - \frac{n}{1-n} v.
\]

Eqs (1) and (2) are continuity equations of the liquid phase and the solid phase, and Eqs (3) and (4) are equations of motion of the liquid phase and the solid phase. The steady state form of Eq. (3) is Darcy’s law.

Let the considered suspension be placed in a vessel with an impervious bottom, which is the case we will study below, and let the initial height of the suspension column be \( L \). The considered domain is then
\[ \Omega = \{ \forall x \in (0, L) \}, \]  
\[ \text{and the boundary conditions imposed at } x = 0 \text{ on } w \text{ and } v \text{ are} \]  
\[ w(0, t) = 0 \text{ and } v(0, t) = 0 \text{ for } t > 0. \]  
\[ \text{From Eqs (1) and (2) we get a simple differential equation} \]  
\[ \frac{\partial(w + v)}{\partial x}(x, t) = 0, \]  
which can be easily solved. All its solutions are \[ w(x, t) + v(x, t) = f(t), \]  
where \( f \) is any differentiable function. To satisfy conditions (7), \( f \) has to be zero. Hence \[ w(x, t) = -v(x, t) \text{ for } (x, t) \in [0, L] \times [0, \infty), \]  
Making use of Eqs (5) and (8) and excluding the derivative \( \frac{\partial p}{\partial x} \) from Eqs (3) and (4), the set of governing equations can be reduced to the continuity Eq. (1)  
\[ \frac{\partial n}{\partial t}(x, t) + \frac{\partial w}{\partial x}(x, t) = 0, \]  
and the equation of motion  
\[ \frac{\partial w}{\partial t} + \frac{n}{\rho_w + n(\rho_s - \rho_w)} \frac{\partial \tau}{\partial x} = \frac{g(\rho_s - \rho_w)n(1 - n)}{\rho_w + n(\rho_s - \rho_w)} \]  
\[ - \frac{g \rho_w n}{(1 - n)K(\rho_w + n(\rho_s - \rho_w))} w. \]  
These equations contain two unknown functions, \( n \) and \( w \), all the other functions are supposed to be given parameters. Particularly the hydraulic conductivity must be known, when solving problems with Eqs (1) and (9). In this paper, we focus on the problem of determining hydraulic conductivity of kaolin suspensions. The suspension concentration defined as mass of the solid phase contained in unit volume of the suspension, i.e. \[ c = \rho_s (1 - n), \]  
is generally used to describe the state of a suspension. In order to respect this, we replace porosity \( n \) by concentration \( c \) in the following text.

We will suppose that under conditions of monotonous processes, hydraulic conductivity \( K \) and solid-phase stress \( \tau \) are functions of the suspension’s concentration. The notion of the monotonous process was introduced by Mls (1995) in order to exclude a possible effect of hysteresis: a process in a suspension is monotonous when it satisfies the condition  
\[ \frac{\partial c}{\partial t}(m_1, t_1) \frac{\partial c}{\partial t}(m_2, t_2) \geq 0, \]  
for any couple of values \( m_1 \) and \( m_2 \) of a material coordinate and for any couple of instants \( t_1 \) and \( t_2 \).

THE MATERIAL OF THE SUSPENSION

The experimental research was carried out with suspensions of water and kaolin. Among the available materials, the Sedlec kaolin Zettlitz Ia was chosen as it is a well defined material making it possible to repeat the experiments at any time later. Moreover, it is also suitable because of its well-balanced granulometric curve, see Fig. 1 and Table 1. The particle size distribution was measured with the Sedigraph ET 5000, manufactured by the Micromeritics. The sedigraph measures particle mass directly via X-ray absorption and determines the equivalent spherical diameter of particles from the rate at which particles fall under gravity through a liquid having known properties. The data presented in Fig. 1 and Table 1 are affected by the accuracy of the described method of measurement. The applied water was deionized before it was used in suspensions.

Kaolinite is a layered silicate mineral with one tetrahedral sheet of SiO₂ linked through oxygen atoms to one octahedral sheet of Al₂O₃. Within the kaolinite structure, there are isomorphic substitutions possible: Silicate (4+) can be surrogated by Al(3+) or by Fe(3+). If so, kaolinite particle carries negative charge on its surface, van Olphen (1963). Table 2 shows the composition of the applied kaolin.
was necessary to measure the position of this interface with a good accuracy and for a relatively wide range of suspension concentrations, a precise rate of coagulation was defined. At any applied suspension concentration, the coagulation was achieved by adding Calcium(II) chloride at 0.4 percent of the solid-phase mass.

### Table 2. Composition of the Kaolin Zettlitz Ia.

<table>
<thead>
<tr>
<th>SiO₂</th>
<th>Al₂O₃</th>
<th>Fe₂O₃</th>
<th>TiO₂</th>
<th>CaO</th>
<th>MgO</th>
<th>K₂O</th>
<th>Na₂O</th>
<th>Loss on ignition</th>
</tr>
</thead>
<tbody>
<tr>
<td>47.28</td>
<td>35.78</td>
<td>0.97</td>
<td>0.26</td>
<td>0.29</td>
<td>0.23</td>
<td>1.05</td>
<td>0.03</td>
<td>14.02</td>
</tr>
</tbody>
</table>

Vertical cylinders with impervious bottom were used to carry out the settling experiments. The height of the cylinders was 2000 mm and, in most cases, their inner diameter was 104 mm. Several cylinders having inner diameters 94 and 114 mm were also used in order to check whether the results were affected by the cross-section surface and perimeter.

### THE LABORATORY MEASUREMENTS

A set of 45 batch tests has been carried out in the laboratory of Department of Hydrogeology at Charles University. The applied suspensions were mixtures of the above described kaolin and water in the precisely defined state of coagulation. A homogeneous mixture was prepared in a tank and was poured into the vessel used for the measurements immediately that after. According to this procedure, we supposed that each test started from homogeneous state of the suspension column. Hence the initial conditions of an experiment are

\[ c(x,0) = c_0 \text{ and } w(x,0) = 0 \quad \text{for } x \in (0,L), \]

where \( L \) is the initial height and \( c_0 \) is the initial concentration of the suspension column at the experiment. A visible interface develops in the suspension during the sedimentation process.

The interface separates the zone of suspension from the overlying layer of water and moves downwards starting at time \( t = 0 \) from the level \( x = L \). We will denote with \( Z(t) \) its height above the bottom at time \( t \). The thickening of the suspension begins at the impervious bottom where the particles of the solid phase are stopped. Consequently, another interface starts its motion from the bottom upwards, separating the layer of suspension of the initial concentration \( c_0 \) from the zone of thickening, see Fig. 2. As the upper interface is clearly visible, it was possible to measure the time dependence of the height of the suspension column. In this way, 45 batch tests were carried out each one for particular initial values of concentration \( c_0 \) and height \( L \).

The data obtained during a test are of the form

\[ \left\{ (t_i, z_i) \right\}_{i=0}^{N_j} \]

where \( z_i \) is the height of the suspension column at time \( t_i \), \( t_0 = 0 \), \( z_0 = L_j \), \( L_j \) is the initial height of the column at \( j \)-th test and \( N_j \) is the number of measurements made during the \( j \)-th test. The first part of a data set obtained during one of the batch tests is depicted by square marks in Fig. 3. If we denote \( Y(t) \) the height of the lower interface above the bottom at time \( t \), i.e. the level below which \( c > c_0 \), the zone between the interfaces contains suspension at the initial concentration \( c_0 \):

\[ c(x,t) = c_0 \quad \text{for } t > 0 \quad \text{and} \quad x \in (Y(t),Z(t)), \]

see Fig. 2. We will utilize this knowledge when solving the following forward problem.

### A FORWARD PROBLEM AND ITS SOLUTION

Let us suppose that there is a region below the upper interface where there is the constant concentration \( c_0 \) of the suspension for certain time interval. In virtue of continuity Eq. (1), the volumetric flux density of the liquid phase does not depend on the space coordinate \( x \) and is a function of time only. Hence, the Eq. of motion (9) becomes the following ordinary differential equation

\[
\frac{\partial w}{\partial t} + \frac{g \rho_w \rho_0}{(1-n_0) \rho_0 \rho_w} = \frac{g (\rho_s - \rho_w) n_0 (1-n_0)}{\rho_0 + n_0 (\rho_s - \rho_w)}.
\]
where \( n_0 \) is the initial value of porosity related to the initial concentration \( c_0 \) by Eq. (10). According to conditions (12), the initial condition is
\[
w(0) = 0. \tag{16}
\]
The solution to the problem (15), (16) is the function
\[
w(t) = \rho(1-n_0)^2 K(c_0) \left( 1 - \exp \left( \frac{-gn_i r}{(1-n_0)(1+\rho n_0)K(c_0)} \right) \right)^t \tag{17}
\]
where
\[
\rho = \frac{\rho_s - \rho_w}{\rho_w}.
\]
Denote by \( v \) the speed of solid-phase particles. Then, according to the definition of the volumetric solid-phase flux density, it holds
\[
v = (1-n)v_s \tag{18}
\]
and at the same time
\[
\frac{dZ}{dt} = v, \tag{19}
\]
as the upper interface moves down with the solid-phase particles. The last two equations together with relations (8) and (17) enable us to formulate the following ordinary differential equation for the height of the upper interface
\[
(1-n_0)\frac{dZ}{dt} = -\rho(1-n_0)^2 K(c_0) \left( 1 - \exp \left( \frac{-gn_i r}{(1-n_0)(1+\rho n_0)K(c_0)} \right) \right)^t \tag{20}
\]
supplemented with the initial condition
\[
Z(0) = L, \tag{21}
\]
where \( L \) is the initial height of the suspension column. Then, the solution to the last problem is
\[
Z(t) = L - \rho(1-n_0)K(c_0)t + \frac{\rho(1+\rho n_0)(1-n_0)^2 K(c_0)^2}{gn_0} \tag{22}
\]
Function \( Z \) by Eq. (20) satisfies every one of the 45 data sets (13) provided all the parameters \( L, \rho, n_0 \) and \( K(c_0) \) agree with the parameters of the test and \( t_0(0, \theta) \), where
\[
t_0 = \sup \left\{ \theta \in R^1; Z(t) > Y(t), t \in (0, \theta) \right\}.
\]
Fig. 3 shows measured values of the upper interface elevation and the associated curve \( Z(t) \) by Eq. (20). Splitting of these curves is clearly visible indicating that \( t_0 \) lies close to 30 hours. Also it can be seen from the figure that the derivatives of both curves decrease at the beginning until they become almost constants. While the theoretical function approaches very quickly to its asymptote, in minutes, the measured data obtain the linear form after several hours. This phenomenon was observed in all the experiments and can be explained by an initial instability of the suspension column.

**DETERMINATION OF THE FUNCTION \( K(c) \)**

Every one of the 45 tests started with its own parameters \( L, \rho, n_0 \) and \( K(c_0) \). The only unknowns among them were the values \( K(c_0) \). The remaining values \( L, \rho \), and \( n_0 \) are measurable and were determined. To get the values \( K(c_0) \), the sum \( S(K) \) of squares of differences between the theoretical function \( Z \) and measured data \((t,z)\),
\[
S(K) = \sum_{i=1}^{N}(z_i - Z(t_i))^2, \tag{23}
\]
was minimized, where the integer \( N \) satisfied the condition \( t_N < t_0 \). The problem leads to a nonlinear equation of a third-order polynomial in the unknown \( K(c_0) \) containing moreover an exponential term. Let us denote
\[
\zeta_i = L - z_i, A = \rho(1-n_0) ,
\]
\[
B = \frac{(1-n_0)^2(1+\rho n_0)\rho}{gn_0} ,
\]
\[
e_i = \exp \left( \frac{-gn_i r}{(1-n_0)(1+\rho n_0)K(c_0)} \right),
\]
i.e.
\[
e_i = \exp \left( \frac{-At_i}{BK(c_0)} \right). \tag{24}
\]
The sum of squares of differences (21) is now
\[
S = \sum_{i=1}^{N} \left( \zeta_i - AK(c_0)t_i + B(K(c_0))^2(1-e_i) \right)^2. \tag{25}
\]
Hence, the sum \( S \) is minimized at \( K(c_0) \) satisfying equation
\[
2B^2 \sum_{i=1}^{N} (1-e_i)^2 (K(c_0))^2 - AB \left( \sum_{i=1}^{N} e_i(t_i - e_i) + 3 \sum_{i=1}^{N} t_i(1-e_i) \right) (K(c_0))^2 + \left( A^2 \sum_{i=1}^{N} e_i^2(t_i - e_i) + 2B \sum_{i=1}^{N} \zeta_i (1 - e_i) \right) (K(c_0)) - 4 \sum_{i=1}^{N} t_i \zeta_i (1 + e_i) = 0. \tag{26}
\]
As the function
\[
e(t) = \exp \left( -\frac{At}{BK(c_0)} \right)
\]
converges to zero exponentially with time, the function \( Z(t) \) quickly approaches its asymptote
\[
\phi(t) = L - \rho(1 - n_0)K(c_0)t + \frac{(1 - n_0)(1 + \rho n_0)\rho(K(c_0))^2}{gn_0}.
\]

Hence, we are looking for such a solution of Eq. (24) that is close to the solution of the equation
\[
2B^2 \sum_{i=1}^{N} (K(c_0))^3 - 3AB \sum_{i=1}^{N} t_i (K(c_0))^2 + \left( A^2 \sum_{i=1}^{N} t_i^2 + 2B \sum_{i=1}^{N} \zeta_i \right) K(c_0) - A \sum_{i=1}^{N} t_i \zeta_i = 0.
\]

(25)

It can be shown that for each data set (13) obtained from the 45 experiments, Eq. (25) has exactly one positive root \( \zeta \) and \( \zeta > 0 \). According to this knowledge, the 45 unknown values \( K(c_0) \) were found by the interval bisection method. Fig. 3 shows the data \( (t_i, \zeta_i) \) obtained from one of the batch tests and the function \( Z(t) \) with the corresponding solution \( K(c_0) \).

The splitting of the depicted functions, which is clearly visible in the figure, indicates the time \( t_m \) at which \( Z(t_m) = Y(t_m) \) and both the interfaces meet. It is also evident that, when looking for \( K(c_0) \), only those elements \( (t_i, \zeta_i) \) of the set (13) can be utilized that satisfy the condition \( t_i < t_m \). It was found that the starting value \( \zeta \) is very close to the solution \( K(c_0) \), i.e. that the linear function
\[
\zeta(t) = L - \rho(1 - n_0)Kt + \frac{(1 - n_0)(1 + \rho n_0)\rhoK^2}{gn_0}
\]
fits the measured data almost as well as the function (20).

Connecting the obtained values \( \{K(c_0)\}_i \), \( i = 1, \ldots, 45 \) with the starting conditions of the \( i \)-th test, particularly with values \( (c_0)_i \), a new data set was obtained:
\[
\{c_i, K_i\}_{i=1}^{45},
\]
(26)

where the notation was simplified writing \( c_i \) and \( K_i \) instead of \( (c_0)_i \) and \( \{K(c_0)\}_i \), respectively. Analyzing the data (26) it was realized that the hydraulic conductivity decreases with concentration in the whole region and that the nature of the decrease is of a power form for lower values of concentration and exponential for its higher values. Using again the least squares method for logarithms of hydraulic conductivity and requiring moreover certain degree of smoothness, the unknown function \( K(c) \) was determined. Particularly the sum
\[
S = \sum_{i=1}^{45} \left( \ln(K_i) - \ln((K(c_i)))^2 \right)
\]
was minimized, and \( K \in C^1(D) \) was required, where \( D \) is the domain of the function \( K \). Eventually, the following results were achieved:
\[
D = [4.50, 421],
\]
\[
K = A_1 \times c^{B_1} \text{ for } c \in [4.50, 95.7438],
\]
\[
K = A_2 \times c^{B_2} \text{ for } c \in [95.7438, 421],
\]
where
\[
A_1 = 0.44872, \quad B_1 = -1.8578,
\]
\[
A_2 = 6.0028 \times 10^{-4}, \quad B_2 = 0.98078.
\]

(28)

Figs 4 and 5 show the curve \( K(c) \) in two different scales; the solid curves depict the obtained function \( K(c) \) and the points (the squares) are the values of \( K \) computed from the 45 batch tests using the above described method. The bold line shows the power function and the thin line shows the exponential function. The different scales of the \( c \)-axes in the figures were chosen in order to make visible how the function changes its nature. The vertical dashed lines in the figures denote the precise position of the threshold of separation.

Fig. 4. Function \( K(c) \).

CONCLUSIONS

The hydraulic conductivity of water and kaolin suspension as a function of its concentration was investigated. Instead of measuring the hydraulic conductivity directly, 45 batch tests were carried out and, during each of them, couples of time and height of the suspension column were measured. Making use of Darcian mechanics of two-phase systems, Eq. (1) to (4), it was
possible to find the solution (20) of the direct problem. The solution of the corresponding inverse problem is based on the comparison of the solution (20) with the measured data. As the solution describes the settling process in its initial phase, before the zone of compression reaches the top of the suspension column, the obtained value of hydraulic conductivity corresponds to the initial concentration of the measured test.

![Function K(c).](image)

The hydraulic properties of a suspension strongly depend on its degree of coagulation. To achieve uniqueness of obtained results, the coagulation was precisely defined for every concentration as the rate of added Calcium(II) chloride and the mass of the solid phase. Particularly 4 g of CaCl₂ were added to 1 kg of the solid phase. In this way, the material is considered as different concentrations of one suspension.

Analysis of the results made it possible to determine the dependence of hydraulic conductivity on the suspension concentration i.e. the function $K(c)$. Denoting the domain of measured concentrations $D$, the requirement $KeC^3(D)$ has been satisfied. Moreover, it was find that the dependence changes its nature. A point $ξ$ was determined such that $ξ ∈ D$ and, as the concentration increases, the function $K$ changes at $ξ$ from a power function to an exponential function.

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